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# Heckman–Opdam's Jacobi polynomials for the $BC_n$ root system and generalized spherical functions

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**Abstract**

We prove that the radial part of the Laplacian on the space of generalized spherical functions on the symmetric space $GL(m+n)/GL(m) \times GL(n)$ is the Sutherland differential operator for the root system $BC_n$ and the radial parts of the differential operators corresponding to the higher Casimirs yield the integrals of the quantum Calogero–Moser system. It allows us to give a representation theoretical construction for the three parameter family of Heckman–Opdam's Jacobi polynomials for the $BC_n$ root system.


*Keywords:* Spherical functions; Heckman–Opdam's Jacobi polynomials; The radial part of the differential operators

## 0. Introduction

Functions on a homogeneous space $G/K$ invariant with respect to the left action of $K$ are called spherical functions (or sometimes $K$-spherical). One can also study functions on $G/K$ with values in a representation $V$ of $G$ which are equivariant with respect to the left action of $K$. This more general class of functions may be called vector valued spherical functions. The theory of such functions was developed by Harish-Chandra, Helgason and other authors [5,8,17].

In the case when $G = K \times K$ and $K \subset G$ is the diagonal subgroup, the study of vector valued spherical functions is equivalent to the study of functions on $K$, equivariant with respect to conjugation. When $K$ is reductive over $\mathbb{C}$, the Peter-Weyl

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theorem gives a description of the space of conjugacy equivariant functions as the space spanned by the vector valued characters of $K$. The article [4] deals with such kind of spherical functions in the case when $K = SL(n, \mathbb{C})$. Namely, the Laplace operator on $K$ restricted to the space of spherical functions can be written in terms of coordinates along the maximal torus. If the vector valued equivariant functions take values in $V = S^{\kappa n}\mathbb{C}^n$, then the resulting operator (the so-called radial part of the Laplacian) coincides, up to an obvious conjugation, with the Sutherland differential operator, which is the Hamiltonian of the Calogero–Moser quantum mechanical system for the root system $A_{n-1}$ [1,15,16]. Differential operators on $K$ corresponding to the higher Casimir operators can also be written in terms of coordinates along the maximal torus. These operators are quantum integrals of the Calogero–Moser system. Moreover, Weyl group invariant eigenfunctions of the Sutherland operator can be expressed as vector valued traces of some intertwining operators between some particular representations of $K$. By the results from [6] these eigenfunctions are essentially Jack polynomials for the root system $A_{n-1}$ (up to a Weyl-determinantlike factor).

The main result of this paper is a representation theoretic interpretation (in the spirit of the work [4]) of the three parameter family of $BC_n$ Heckman-Opdam's Jacobi polynomials. More precisely, we consider the case of the pair $G = GL(m + n, \mathbb{C})$ $(m \geqslant n)$, $K = GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$, slightly modify the definition of spherical functions and as result we get a similar to [4] theory for the root system $BC_n$. Namely, in this case the Laplace operator on $G$, written in terms of coordinates of some torus inside $G$, yields the Sutherland operator for the root system $BC_n$, and the higher Casimir operators give quantum integrals of the corresponding quantum mechanical system.

Furthermore, the restriction to the torus of some special matrix elements of irreducible finite-dimensional representations $L_\lambda$ of $G$, where $\lambda$ ranges over the set isomorphic to the cone of dominant integral weights of the root system $C_n$, yields $W$-invariant eigenfunctions of the Sutherland differential operator. The Sutherland differential operator after a suitable gauge transformation becomes operator from the paper [6]. By Heckman and Schlichtkrull [6] $BC_n$ Heckman–Opdam's Jacobi polynomials are Weyl group invariant eigenfunctions of this operator. Thus, we obtain a representation theoretic interpretation of Heckman–Opdam's Jacobi polynomials.

In [7] (see also [2] for compact exposition of the result and for its $q$-analog) a one parameter family of the $BC_n$ Heckman–Opdam's Jacobi polynomials was constructed by means of spherical functions on $G$. This family is a subfamily of the three parameter family of $BC_n$ Heckman–Opdam's Jacobi polynomials from Theorem 2.2 below. In the one dimensional case our construction reduces to the result of the paper [11]. More exactly, the author of [11] consider the quantum version of the pair $(K, G) = (U(1), SU(2))$ and give representation theoretic interpretation for the four-parameter family of so-called Askey–Wilson polynomials. The quasiclassical limit of this family is the two-parameter family from our paper.

The structure of the paper is as follows. In Section 1, we give an interpretation of the results on vector valued characters from [4] using the point of view of the theory of symmetric spaces. Section 2 contains the main result: the construction of Jack polynomials through vector valued twisted spherical functions on the symmetric space $G/K$. The proofs of the claims from Section 2 are given in Section 3. All constructions are easier for the special case $m = n$, $\varkappa_{(1)} + \varkappa_{(2)} = 0$ and to gain a better understanding, the reader is advised to consider this special case separately.

The results of this paper can be generalized to the case of quantum symmetric spaces [14]. The quantum version of the construction yields the five parameter family of Macdonald–Koornwinder polynomials, which are $q$-analog of Heckman–Opdam's Jacobi polynomials.

## 1. Jack polynomials for the $A_{n-1}$ root system

In this section we explain how to interpret the results of the paper [4] from the point of view of the theory of symmetric spaces. This section does not contain any proofs.

### 1.1. The space of spherical functions

Let $K$ be the group $SL(n, \mathbb{C}) = SL(n)$ and $G = K \times K$. The diagonal embedding $K \hookrightarrow K \times K$ gives rise to the left and right action of $K$ on $G$. Let $V(\kappa)$ be the representation $S^{n\kappa}\mathbb{C}^n$ of $K$, $\kappa \in \mathbb{Z}_+$.

The space of $K$-spherical functions $F_\kappa$ is defined by the formula

$$F_\kappa = \{ f \in F(G, V(\kappa)) \,|\, f(kgk') = kf(g), \forall k, k' \in K, g \in G \},$$

where $F(G, V(\kappa))$ is the space of $V(\kappa)$ valued polynomial functions on $G$. We can think about this space as about the space $K$-equivariant functions on the quotient space $G/K$. The quotient space $G/K$ can be identified with the group $K : (x, y) \to xy^{-1}$. That is an embedding $K \hookrightarrow G$, $k \mapsto (k, 1)$ induces isomorphism between $K$ and $G/K$. Under this identification the right action of group $K$ becomes the conjugation action and

$$F_\kappa = \{ f \in F(K, V(\kappa)) \,|\, f(kk'k^{-1}) = kf(k'), \forall k, k' \in K \}.$$

### 1.2. The restriction of spherical functions to a maximal torus and differential operators

Obviously, the restriction of a function from $F_\kappa$ to the maximal torus $H = \{ e^{h(x)} | h(x) = diag(x_1, \dots, x_n), \mathrm{tr}(h(x)) = 0 \}$ takes values in the one dimensional space $V(\kappa)[0]$, that is it can be regarded as a scalar function.

Furthermore, functions from $F_\kappa$ are uniquely determined by their restriction to the maximal torus (because the generic element of $K$ is conjugate to an element of $H$). The Laplace operator on $K$ being restricted to $F_\kappa$ takes the form

$$\bar{L}_\kappa = \delta^{-1} \left( \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \kappa(\kappa + 1) \sum_{i<j} \frac{1}{2\sinh^2((x_i - x_j)/2)} - \frac{n^3 - n}{12} \right) \delta,$$

where $\delta = \prod_{i<j} \sinh((x_i - x_j)/2)$ and $x_i$ $(i = 1, \dots, n)$ are the natural coordinates on $H$. The operator $\delta \bar{L}_\kappa \delta^{-1}$ coincides, up to an additive constant, with the Sutherland operator—the Hamiltonian of the quantum $n$-body system on the line with interaction potential $\kappa(\kappa+1)\sinh^{-2}(y/2)$.

If $C_1, \dots, C_n$ are generators of the center $\mathfrak{Z}(\mathfrak{k})$ of the universal enveloping algebra $U(\mathfrak{k})$ of the Lie algebra $\mathfrak{k}$ of the group $K$, then the corresponding differential operators on $K$ map the space $F_\kappa$ into itself. Hence the restriction of such operator to the space $F_\kappa$ is a uniquely determined differential operator and it is possible to express it in terms of coordinates along the maximal torus. Let us denote such differential operator on the torus by $R_{C_i}$.

### 1.3. The connection with quantum integrable systems

Recall that an operator commuting with the Hamiltonian of a quantum physical system is called a *quantum integral* of this system. Let $L$ be a differential operator in $n$ variables. One says that $L$ defines a *completely integrable quantum Hamiltonian system* if there exists a set of $n$ algebraically independent quantum integrals $L_1, \dots, L_n$ which are differential operators and commute with each other. The collection of operators $L_1, \dots, L_n$ is called a complete system of quantum integrals for $L$.

It is clear from the above that we have following proposition

**Proposition 1.1.** (1) [15] *The Sutherland differential operator defines a completely integrable system; moreover*

(2) [3] *the operators* $\delta R_{C_i} \delta^{-1}$, $i = 1, \dots, n$ *form a complete system of quantum integrals for this system.*

### 1.4. Jack polynomials

Below we give a definition of Jack polynomials following the work [6]. The action of the operator $\widetilde{L}_\kappa = \delta^{-\kappa} \bar{L}_\kappa \delta^\kappa$ on the space $\mathbb{C}[P]^W$ of Weyl invariant Laurent polynomials on $H$ is diagonalizable, and the eigenfunctions have the form $J_\lambda^\kappa = m_\lambda + \sum_{v<\lambda} s_{\lambda v} m_v$, where $m_\lambda(x) = \sum_{v \in W\lambda} e^{(v,x)}$, where $\lambda \in P_+(SL(n)) = \{\lambda \in \mathbb{Z}^n | \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n\}$ is an integral dominant weight, defined up to shift, and $<$ is the standard partial order on weights. Polynomials $J_\lambda^\kappa$ are called *Jack polynomials* for the root system $A_{n-1}$.

### 1.5. The formula for the spherical eigenfunction

Let $L_\lambda$ be the finite-dimensional representation of $K$ with highest weight $\lambda \in P_+(SL(n))$ and $L_\lambda^*$ is its dual. Then the restriction of the finite-dimensional representation $L_\lambda \boxtimes L_\lambda^*$, of $G = K \times K$ to the diagonally embedded subgroup $K$ yields the representation $L_\lambda \otimes L_\lambda^*$ of $K$. The tensor product $L_\lambda \otimes L_\lambda^*$ contains the vector $w_0 = \sum u_i \otimes u_i^*$ (where $\{u_i\}$ is a basis of $L_\lambda$ and $\{u_i^*\}$ is the corresponding dual basis of $L_\lambda^*$), which is stable with respect to the (diagonal) action of $K$. All weight spaces of the representation $V(\kappa)$ are one dimensional. Thus, elementary calculation with the Weyl character formula shows that if $\lambda = \mu + \kappa\rho$, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$, $\mu \in P_+(SL(n))$, then $L_\lambda \otimes L_\lambda^*$ also contains a unique copy of the irreducible representation $V(\kappa)$.

Thus for the tensor product one can write a decomposition

$$L_\lambda \otimes L_\lambda^* = V(\kappa) \oplus \bigoplus_{\mu \in C_\lambda} L_\mu,$$

$C_\lambda \subset P_+(SL(n))$, $V(k) \neq L_\mu$ for all $\mu \in C_\lambda$. Hence a vector $u \in L_\lambda \otimes L_\lambda^*$ can be uniquely presented in the form $u = \widetilde{u} + \sum_{\mu \in C_\lambda} u_\mu$, where $\widetilde{u} \in V(\kappa)$, $u_\mu \in L_\mu$. This decomposition gives an embedding $s : V^*(\kappa) \hookrightarrow L_\lambda^* \otimes L_\lambda$, $s(x)(u) = x(\widetilde{u})$.

Let $v_1, \dots, v_N$ be a basis of representation $V(\kappa)$. Then one can construct the function

$$\Psi_\mu(g) = \sum v_i \langle s(v_i^*), g w_0 \rangle$$

(where $v_i^*$, $i = 1, \dots, N$ is a basis dual to the basis $v_i$, $i = 1, \dots, N$). This function belongs to the space $F_\kappa$. Indeed, right $K$-invariance holds because $w_0$ is $K$-invariant and left $K$-equivariance follows from the independence of $\Psi_\mu$ from the choice of the basis $v_i$, $i = 1, \dots, N$ in $V(\kappa)$. The maximal torus of $K$ is embedded in $G$ by map $e^{h(x)} \mapsto (e^{h(x)}, 1)$. Because of independence of $\Psi_\mu$ from the choice of the basis $v_i$, $i = 1, \dots, N$ we get that the restriction of $\Psi_\mu$ to this torus has the form

$$\Psi_\mu(x) = w_\kappa \langle s(w_\kappa^*), e^{h(x)} w_0 \rangle,$$

where $w_\kappa^*$, $w_\kappa$ are the vectors such that $V(\kappa)^*[0] = span\{w_\kappa^*\}$, $V(\kappa)[0] = span\{w_\kappa\}$ and $\langle w_\kappa, w_\kappa^* \rangle = 1$.

Let us identify $V(\kappa)[0]$ with $\mathbb{C}$ via $w_\kappa \to \langle w_\kappa, v_\lambda^* \otimes v_\lambda \rangle$, where $v_\lambda$ and $v_\lambda^*$ are the highest and lowest weight vectors for the representations $L_\lambda$ and $L_\lambda^*$, respectively. Under this identification function $\Psi_\mu$ becomes the element of the ring $\mathbb{C}[P] = span\{e^{(\mu,x)}, \mu \in \mathbb{Z}^n, \sum \mu_i = 0\}$ (here we use notation $(\cdot,\cdot)$ for the standard scalar product on $\mathbb{C}^n$).

## 1.6. The main theorem

An element of the center of the universal enveloping algebra $U(\mathfrak{k} \oplus \mathfrak{k})$ acts on the space $L_\lambda \boxtimes L_\lambda^*$ by a constant. In particular, the element $C_i \in U(\mathfrak{k})$, embedded into $U(\mathfrak{k} \oplus \mathfrak{k})$ via $x \mapsto (x, 0)$ $(x \in \mathfrak{k})$, does. Hence $\Psi_\mu(x)$ is an eigenfunction of $R_{C_i}$. Obviously, $\Psi_\mu(x)$ is $W$-invariant when $\kappa$ is even and $W$-antiinvariant when $\kappa$ is odd. Moreover, the following theorem holds:

**Theorem 1.1** (Etingof et al. [4]). *Under the above identification of* $V(\kappa)[0]$ *with* $\mathbb{C}$:

(1) $\Psi_0(x) = \delta(x)^\kappa$.
(2) $\Psi_\mu(x)$ *is divisible by* $\Psi_0(x)$ *in algebra* $\mathbb{C}[P]$ *for all* $\mu \in P_+(SL(n))$.
(3) $\Psi_\mu(x)/\Psi_0(x) = J_\mu^\kappa(x)$.

## 2. Heckman–Opdam's Jacobi polynomials for the $BC_n$ root system

In this section we explain a representation theoretic construction of the three parameter family of Heckman–Opdam's Jacobi polynomials for the root system $BC_n$. We postpone all proofs (which are mostly lengthy calculations) for the next section.

### *2.1. The symmetric pair $(G, K)$ and restricted root system*

In this section we introduce notations for representation theoretic objects which are necessary for a further exposition.

Let $G$ be the group $GL(m+n, \mathbb{C}) = GL(m+m)$ and $\mathfrak{g}$ its Lie algebra, where $m \geqslant n$. The conjugation by the diagonal matrix $J \in G$, $J_{ii} = 1$, $i = 1, \dots, m$, $J_{jj} = -1$, $j = m+1, \dots, m+n$ defines an involution $\Theta$: $\Theta(g) = JgJ^{-1}$. $\Theta$-invariant elements of $G$ form a subgroup $K = K_{(1)} \times K_{(2)} = GL(m) \times GL(n) \subset G$. The differential of this involution at the unit acts on the Lie algebra $\mathfrak{g}$ and induces the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} = \{x \in \mathfrak{g} | d\Theta(x) = x\}$, $\mathfrak{p} = \{x \in \mathfrak{g} | d\Theta(x) = -x\}$ ($\mathfrak{k}$ is exactly the Lie algebra of the group $K$). The subspace $\mathfrak{p} \subset \mathfrak{g}$ is not a subalgebra of $\mathfrak{g}$ but nevertheless one can fix a maximal abelian subalgebra $\mathfrak{a}$ inside $\mathfrak{p}$. We use notation $A$ for the corresponding abelian subgroup. Below we fix some particular subgroup $A$ which we will work with.

Before fixing the choice of $A$ we introduce the Cartan subgroup $H$ of the group $G$. The subgroup $H$ is conjugated to the subgroup of the diagonal matrices by matrix $\widetilde{J}$. Matrix $\widetilde{J}$ consists of the diagonal blocks: $\widetilde{J}_{ii} = \widetilde{J}_{m+i,i} = \widetilde{J}_{i+m,i+m} = -\widetilde{J}_{i,i+m} = \frac{1}{\sqrt{2}}$, $i = 1, \dots, n$, $\widetilde{J}_{j+n,j+n} = 1$, $j = 1, \dots, m-n$ and all other entries are zero:

$$\widetilde{J} = \begin{pmatrix} \frac{I_n}{\sqrt{2}} & 0 & -\frac{I_n}{\sqrt{2}} \\ 0 & I_{m-n} & 0 \\ \frac{I_n}{\sqrt{2}} & 0 & \frac{I_n}{\sqrt{2}} \end{pmatrix}.$$

We denote by the symbol $h(x, y, z)$ an element of $H$ of the form $\widetilde{J}e^{diag(x,y,z)}\widetilde{J}^{-1}$, $x, z \in \mathbb{C}^n, y \in \mathbb{C}^{m-n}$. Let $\mathfrak{h}$ be a Lie algebra of $H$.

We put $A$ to be equal to $\exp(\mathfrak{p}) \cap H$ or $A = \{h(x, 0, -x), x \in \mathbb{C}^n\}$. We use notation $e^{a(x)} = h(x, 0, -x)$, $x \in \mathbb{C}^n$ for the elements of $A$, where $a(x) = \sum_{i=1}^n x_i(E_{i+m,i} - E_{i,i+m}) \in \mathfrak{g}$, $E_{ij}$ is the notation for the $ij$-th matrix unit (basis in $\mathfrak{g}$).

The inclusion $a:\mathfrak{a}\hookrightarrow\mathfrak{h}$ induces the projection $\mathfrak{h}^*\twoheadrightarrow\mathfrak{a}^*$. The root system $R\subset\mathfrak{h}^*$ is mapped under this projection onto the restricted root system. The restricted root system is isomorphic to the root system $C_n$, in the case $n=m$, and to the root system $BC_n$, in the case $m>n$. We use the notation $\Sigma$ for the root system $BC_n$ (or $C_n$ when $n=m$). The short, medium and long positive roots of $\Sigma$ are the vectors:

$$\varepsilon_i(1\leqslant i\leqslant n),\quad \varepsilon_i\pm\varepsilon_j(1\leqslant i<j\leqslant n),\quad 2\varepsilon_i(1\leqslant i\leqslant n), \tag{1}$$

where $\varepsilon_i(a(x))=x_i$. The multiplicity of the root $\alpha\in\Sigma$ is the number of the roots from $R$ projecting into the root $\alpha$. The root multiplicities for the short, medium and long roots of $\Sigma$ are $t_1=2(m-n)$, $t_2=2$, $t_3=1$. Below we use the half multiplicities $s_i=t_i/2$, $i=1,2,3$. The bilinear form $(\cdot,\cdot)$ on $\mathfrak{a}^*$ is the standard one: $(\varepsilon_i,\varepsilon_j)=\delta_{ij}$.

Let us also introduce notations for a generalization of the Weyl denominator:

$$\delta_{p_1,p_2,p_3}(x)=\prod_{\alpha\in\Sigma_+}\sinh^{p_\alpha}(\alpha(a(x))),$$

and for the vector:

$$\rho_{p_1,p_2,p_3}=\frac{1}{2}\sum_{\alpha\in\Sigma_+}p_\alpha\alpha,$$

where $p_{\pm\varepsilon_i}=p_1$, $p_{\pm\varepsilon_i\pm\varepsilon_j}=p_2$ $(1\leqslant i<j\leqslant n)$, $p_{\pm2\varepsilon_i}=p_3$ $(1\leqslant i<j\leqslant n)$. Below we always use the convention that for the given vector $\vec{p}\in\mathbb{C}^3$, $p_\alpha$ means the same thing as in the previous formula if $\alpha\in\Sigma$ and $p_\alpha=0$ if $\alpha\notin\Sigma$.

### 2.2. Vector valued spherical functions

In this subsection we define the space of $K$-equivariant twisted vector valued functions on $G$. Functions from this space take values in the space of the particular representation of $K$. In principle one has a lot of freedom in the choice of this representation and it is not clear why the chosen representation is better than any others. We will explain it in the next subsection.

First we define the space of ''twisted'' scalar valued functions on $G/K$:

$$\hat{F}=\{f\in F|f(gk)=f(g)det^{\varkappa_{(1)}}(k_{(1)})det^{\varkappa_{(2)}}(k_{(2)}),\forall k\in K,g\in G\},$$

where $F$ is the space of polynomial functions on $G$ and $k=k_{(1)}k_{(2)}$, $k_{(i)}\in K_{(i)}$ (let us remind that $K_{(1)}=GL(m)$, $K_{(2)}=GL(n)$). The desired space of spherical vector valued $K$-spherical functions is a subspace of the tensor product $\hat{F}\otimes U_{\vec{\varkappa}}$ where $U_{\vec{\varkappa}}$ is a representation of $K$ which we define below.

Let us fix notations for finite-dimensional representations of $GL(r)$. A finite dimensional irreducible representation of $GL(r)$ is encoded by its highest weight $\lambda\in\mathbb{Z}^r$, $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_r$, which is dominant and integral. We denote the corresponding representation by $L_\lambda$ and the set of highest weights by $P_+(GL(r))$. The determinant representation in our notations is $L_{1^r}$, $1^r$ is the $r$-dimensional vector consisting of ones, $1^r = (1, \dots, 1)$.

The representation $U_{\vec{\varkappa}}$ of the group $K = K_{(1)} \times K_{(2)} = GL(m) \times GL(n)$ is of the form $U_{\vec{\varkappa}} = W_{\vec{\varkappa}} \boxtimes V_{\vec{\varkappa}}$ and element $k = k_{(1)}k_{(2)} \in K$ acts on $u = w \otimes v \in U_{\vec{\varkappa}}$ by formula $k(u) = k_{(1)}(w) \otimes k_{(2)}(v)$.

The representation $W_{\vec{\varkappa}}$ is an irreducible representation $L_\mu$, $(\widetilde{\varkappa}_{(1)} - \varkappa_{(1)})1^n + \varkappa_{(1)}1^m$ of the group $K_{(1)} = GL(m)$. Here $\widetilde{\varkappa}_{(1)}, \varkappa_{(1)}$ are integers and $1^n \in P_+(GL(m))$ is a vector of the form $(1, \dots, 1, 0, \dots, 0)$ with ones at first $n$ places and zeroes at other places. Why we choose such representation is clear from Lemma 2.1.

The representation $V_{\vec{\varkappa}}$ is an irreducible representation $L_\lambda \otimes det^{\varkappa_{(2)}+\varkappa_{(1)}-\widetilde{\varkappa}_{(1)}}$ of $K_{(2)} = GL(n)$, where $\lambda = (\varkappa_v(n-1), -\varkappa_v, \dots, -\varkappa_v)$. More explicitly, the representation $L_\lambda$ is the representation $S^{\varkappa_v n}\mathbb{C}^n$ (i.e. $\varkappa_v n$ symmetric power of the vector representation) of $PGL(n)$ pulled back to $GL(n)$ i.e. the center acts trivially. The main reason why we choose this representation is that zero weight space of $L_\lambda$ is one dimensional. Below we use notation $\vec{\varkappa} = (\varkappa_{(1)}, \varkappa_{(2)}, \widetilde{\varkappa}_{(1)}, \varkappa_v)$.

**Remark 2.1.** In the simplest case, when $m = n$, $W_{\vec{\varkappa}} \simeq det^{\widetilde{\varkappa}_{(1)}}$ is one dimensional. So $U_{\vec{\varkappa}}$ becomes the representation $det^{\widetilde{\varkappa}_{(1)}} \boxtimes (det^{-\widetilde{\varkappa}_{(1)}} \otimes L_\lambda)$.

Combining all components we get a definition of the space of $K$-equivariant vector valued twisted spherical functions:

$$F_{\vec{\varkappa}} = \{ f \in \widetilde{F} \otimes U_{\vec{\varkappa}} | f(kg) = kf(g), \forall k \in K, g \in G \}.$$

For brevity we call the functions from this space spherical functions.

**Remark 2.2.** In the case $\varkappa_v = 0$, $\varkappa_{(1)} = \widetilde{\varkappa}_{(1)}$, this space was studied in the first part of the book [7].

### 2.3. Properties of the spherical functions

In this subsection we explain why the restriction of a spherical function on the torus $A$ is a scalar function.

Elementary arguments from the linear algebra show that the generic element $g$ of $G$ can be presented in the form $g = ke^{a(x)}k'$, $k, k' \in K$, $x \in \mathbb{C}^n$ and this decomposition is unique up to the action of the Weyl group (this is the particular case of $KAK$-decomposition theorem see lecture 2 the second part of the book [7]). Because of the bi-$K$-equivariance of the functions from $F_{\vec{\varkappa}}$, any function $f \in F_{\vec{\varkappa}}$ is uniquely determined by its restriction to $A$.

The element $y$ of the group $M = Z_K(A) = GL(m-n)$ acts on a spherical function $f$ following way:

$$yf(e^{a(x)}) = f(ye^{a(x)}) = f(e^{a(x)}y) = f(e^{a(x)})det(y)^{\varkappa_{(1)}}.$$

That is the restriction of a spherical function takes values in the subspace $\widetilde{W}_{\vec{\varkappa}} \boxtimes V_{\vec{\varkappa}}$, where $\widetilde{W}_{\vec{\varkappa}}$ is a subspace on which $M$ acts by the character $det^{\varkappa_{(1)}}$.

**Lemma 2.1.** *The subspace $\widetilde{W}_{\vec{\varkappa}}$ is one dimensional. The subgroup*

$$\widetilde{K}_{(1)} = Z_{K_{(1)}}(M) = GL(n)$$

*acts on $\widetilde{W}_{\vec{\varkappa}}$ by the character $det^{\widetilde{\varkappa}_{(1)}}$.*

The proof of the lemma is given at the Section 3.

**Remark 2.3.** In the simplest case $m = n$, the lemma is trivial, since then $W_{\vec{\varkappa}}$ reduces to the one dimensional representation $det^{\widetilde{\varkappa}_{(1)}}$ of $K_{(1)} = \widetilde{K}_{(1)}$.

Now let $T \subset K$ be the subgroup $T = K \cap H = \{k \in K | k = h(x, z, x), x \in \mathbb{C}^n, z \in \mathbb{C}^{m-n}\}$. Observe that $T \subset \widetilde{K}_{(1)} \times M \times K_{(2)}$ and elements of $T$ commutes with $A$. An element $h_2 = e^{diag(0,0,x)}$, $x \in \mathbb{C}^n$ of the Cartan subgroup of $K_{(2)}$ can be presented in the form $h_2 = th_1$, where $h_1 = e^{diag(-x,0,0)} \in \widetilde{K}_{(1)}$ and $t \in T$. Hence using $det(h_1)det(h_2) = 1$ one gets for a spherical function $f$:

$$h_2 f(e^{a(x)}) = f(h_1 t e^{a(x)}) = h_1 f(e^{a(x)}) t = det(h_2)^{\varkappa_{(1)} + \varkappa_{(2)} - \widetilde{\varkappa}_{(1)}} f(e^{a(x)}).$$

All weight spaces of the representation $S^{\varkappa_v n}\mathbb{C}^n$ are one dimensional hence the same is true about $V_{\vec{\kappa}} = S^{\varkappa_v n}\mathbb{C}^n \otimes det^{\varkappa_{(2)} + \varkappa_{(1)} - \widetilde{\varkappa}_{(1)}}$. That means that the restriction of $f$ to $A$ takes values in the one dimensional space $\widetilde{U}_{\vec{\varkappa}} = \widetilde{W}_{\vec{\varkappa}} \boxtimes \widetilde{V}_{\vec{\varkappa}}$, with $\widetilde{V}_{\vec{\varkappa}} = V_{\vec{\varkappa}}[(\varkappa_{(1)} + \varkappa_{(2)} - \widetilde{\varkappa}_{(1)})1^n] \simeq \mathbb{C}$. That is, the restriction of a spherical function $f$ to $A$ is a scalar valued function.

### *2.4. The center of $U(\mathfrak{g})$ and radial parts of biinvariant differential operators*

In this subsection we explain the correspondence between elements of the center of the universal enveloping algebra and the Weyl group invariant differential operators on $A$. This correspondence is given by the radial parts of the differential operators. Finally, we calculate the radial part of the Casimir operator.

The universal enveloping algebra $U(\mathfrak{g})$ may be identified with the algebra of the left $G$-invariant differential operators on $G$. Namely, the element $x \in \mathfrak{g}$ gives the differential operator $D_x f(g) = \frac{d}{dt} f(ge^{tx})|_{t=0}$ and this map can be extended to $U(\mathfrak{g})$: $D_{xy} f = D_x D_y f$, for $x, y \in U(\mathfrak{g})$. A differential operator corresponding to an element of the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ is bi-$G$-invariant, hence it preserves the space $F_{\vec{\varkappa}}$. As any function from $F_{\vec{\varkappa}}$ is uniquely determined by its restriction to $A$, the differential operator $D_C$, $C \in \mathfrak{Z}(\mathfrak{g})$ can be written in terms of coordinates along $A$, and the resulting operator is a differential operator with coefficients in $End(\widetilde{U}_{\vec{\varkappa}}) = \mathbb{C}$. We call this expression the *radial part of* $D_C$ on $F_{\vec{\varkappa}}$ and denote it $R_C$.

**Remark 2.4.** In reality, not any function on the torus $A$ is the restriction of a $K$-biequivariant function. Later we will show that the restriction of the functions from $F_{\vec{\varkappa}}$ span the space of Laurent polynomials of $e^{2x_i}$ ($x_i$, $i = 1, \ldots, n$, are the coordinates along the torus $A$) satisfying vanishing conditions at zero locus of Weyl determinant (see Lemma 2.5) and condition that $W$ acts by the character $\chi$ (see Lemma 2.4). Here $W$ is the Weyl group for the $BC_n$ root system and $\chi$ is the $\mathbb{Z}_2$-character of $W$. But the standard reasoning (see for example [7, p. 16]) shows that a differential operator is uniquely determined by its action on the space of $\chi$-$W$-invariant Laurent polynomials, hence the radial part of $D_C$ is uniquely defined.

The center $\mathfrak{Z}(\mathfrak{g}) \subset U(\mathfrak{g})$ contains the Casimir $C_2 = \sum_{1 \leqslant i,j \leqslant n+m} E_{ij} E_{ji}$. The radial part $R_{C_2}$ can be calculated explicitly. Below we use following notations $\vec{\kappa} = (\kappa_1, \kappa_2, \kappa_3) = (\varkappa_{(2)} - \varkappa_{(1)}, \varkappa_v, \widetilde{\varkappa}_{(1)} - \varkappa_{(2)})$, $\vec{r} = \vec{\kappa} + \vec{s}$. We consider only the case when $\vec{\kappa}$ satisfies the following condition:

$$\kappa_3 \geqslant \kappa_1 + \kappa_3 \geqslant 0. \tag{2}$$

**Theorem 2.1.** *The second order differential operator* $R_{C_2}$ *has the form*

$$2(R_{C_2}\psi)(x) = \delta_{\vec{s}}^{-1}(x)(\Delta_A - u_{\vec{r}}(x) + C_{\vec{\varkappa}})\delta_{\vec{s}}(x)\psi(x), \tag{3}$$

$$C_{\vec{\varkappa}} = \frac{m+n-(m+n)^3}{6} + \frac{(m-n)^3 - m + n}{6} + (\varkappa_{(1)} + \varkappa_{(2)})^2 n + 2(m-n)\varkappa_{(1)}^2,$$

$$u_{\vec{r}}(x) = \sum_{\alpha \in \Sigma_+} \frac{r_\alpha(r_\alpha + 2r_{2\alpha} - 1)(\alpha, \alpha)}{\sinh(\alpha(a(x)))^2},$$

*where* $\Delta_A = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ *is the Laplace operator.*

The operator $L_{\vec{r}}^{\mathrm{CM}} = \Delta_A - u_{\vec{r}}(x)$ is called the Calogero–Moser operator for the root system $BC_n$. The proof of this theorem is given in the next section.

**Remark 2.5.** When $\varkappa_{(1)} + \varkappa_{(2)} = 0$, $m = n$, the coefficient $C_{\vec{\varkappa}}$ is equal to $-2(\rho, \rho)$ ($\rho$ is half sum of the positive roots of the root system $A_{m+n-1}$).

### 2.5. The spherical representations of $G$

In this subsection we describe the finite-dimensional representations of $G$ containing the representation $U_{\vec{\varkappa}}$ of $K$. Proofs of the statements use the Littlewood–Richardson rule and are given at the next section.

First we introduce some new definitions and notations. Let us denote by $P^{\varkappa_{(1)},\varkappa_{(2)}}$, $\varkappa_{(1)},\varkappa_{(2)}\in\mathbb{Z}$, ($\varkappa_{(1)}\geqslant\varkappa_{(2)}$ by (2)) the subset of $P_+(GL(n+m))$ consisting of $\lambda\in P_+(GL(n+m))$ such that $G$-representation $L_\lambda|_K$ contains a copy of $det^{\varkappa_{(1)}}\boxtimes det^{\varkappa_{(2)}}$.

**Lemma 2.2.** *$\lambda\in P^{\varkappa_{(1)},\varkappa_{(2)}}$ if and only if*

$$\lambda_j+\lambda_{m+n+1-j}=\varkappa_{(1)}+\varkappa_{(2)}\quad (j=1,\dots,n),$$

$$\lambda_{n+j}=\varkappa_{(1)}\quad (j=1,\dots,m-n),$$

$$\lambda_n\geqslant\varkappa_{(1)}.$$

*Moreover, if $\lambda\in P^{\varkappa_{(1)},\varkappa_{(2)}}$ then $L_\lambda|_K$ contains a unique copy of the representation $det^{\varkappa_{(1)}}\boxtimes det^{\varkappa_{(2)}}$.*

**Remark 2.6.** In the case when all $\varkappa_{(i)}$ are zero the last lemma follows from the fact that $(G,K)$ is a symmetric pair (see [9, Chapter V, Theorem 4.1]).

We denote by the symbol $P_+^{BC}$ the set of $n$-tuples of non-negative integers $\mu\in\mathbb{Z}_+^n$ such that $\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_n\geqslant 0$. Now consider the map $\tau$: $P_+^{BC}\to\mathbb{Z}^{m+n}$, where

$$\tau(\mu)=(\mu_1+\hat{\varkappa},\dots,\mu_n+\hat{\varkappa},\varkappa_{(1)},\dots,\varkappa_{(1)},\hat{\varkappa}-\mu_n,\dots,\hat{\varkappa}-\mu_1),$$

$\frac{\varkappa_{(1)}+\varkappa_{(2)}}{2}=\hat{\varkappa}$. Lemma 2.2 says that $\lambda\in P^{\varkappa_{(1)},\varkappa_{(2)}}$ if and only if $\lambda=\tau(\rho_{-\kappa_1,0,0}+\mu)$ for some $\mu\in P_+^{BC}$. That is, Lemma 2.2 implies that the set $P^{\varkappa_{(1)},\varkappa_{(2)}}$ is isomorphic to $P_+^{BC}$.

**Definition 1.** A finite-dimensional irreducible representation $L_\lambda$ of $G$ is called $\vec{\varkappa}$-spherical ($\vec{\varkappa}$ is integer and satisfies (2)), if $\lambda\in P^{\varkappa_{(1)},\varkappa_{(2)}}$ and $L_\lambda|_K\supset U_{\vec{\varkappa}}$. Let us denote by $P^{\vec{\varkappa}}$ a set of weights $\lambda$ such that $L_\lambda$ is $\vec{\varkappa}$-spherical.

**Lemma 2.3.** *If $\mu\in P_+^{BC}$ then $\lambda=\tau(\mu+\rho_{\vec{\kappa}})\in P^{\vec{\varkappa}}$, and in this case $L_\lambda|_K$ contains a unique copy of the representation $U_{\vec{\varkappa}}$.*

Also Lemma 2.2 and (2) imply that $\tau(\mu+\rho_{\vec{\kappa}})\in P^{\varkappa_{(1)},\varkappa_{(2)}}$ for $\mu\in P_+^{BC}$.

**Remark 2.7.** The statement converse to the Lemma 2.3 also holds. It follows from the statements that are stated below (see Corollary 2). Let us also remark that in the simplest case ( $n=m$, $\varkappa_{(1)}+\varkappa_{(2)}=0$) the map $\tau$ does not depend on $\vec{\varkappa}$.

*2.6. Spherical functions through $\vec{\varkappa}$-spherical representations*

For $\lambda \in P^{\vec{\varkappa}}$ we have a decomposition

$$L_\lambda|_K = U_{\vec{\varkappa}} \bigoplus_{(\mu,\mu') \in C_\lambda} L_\mu \boxtimes L_{\mu'},$$

$C_\lambda \subset P_+(GL(m)) \oplus P_+(GL(n))$. Hence a vector $v \in L_\lambda$ can be uniquely presented in the form $v = \tilde{v} + \sum_{(\mu,\mu') \in C_\lambda} v_{\mu,\mu'}$, $\tilde{v} \in U_{\vec{\varkappa}}$, $v_{\mu,\mu'} \in L_\mu \boxtimes L_{\mu'}$. It allows us to define an embedding $s : U^*_{\vec{\varkappa}} \hookrightarrow L^*_\lambda|_K$, $s(x)(v) = x(\tilde{v})$.

Let $u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda) \in L_\lambda$, $\lambda \in P^{\varkappa_{(1)},\varkappa_{(2)}}$ be a vector such that

$$span\{u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda)\} = det^{\varkappa_{(1)}} \boxtimes det^{\varkappa_{(2)}}.$$

The vector $u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda)$ is unique up to normalization. Now consider the function $\Psi_\mu : G \to U_{\vec{\varkappa}}$, $\mu \in P_+^{BC}$, given by

$$\Psi_\mu(g) = \sum v_i \langle s(v_i^*), g u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda) \rangle,$$

where $\lambda = \tau(\mu + \rho_{\vec{\varkappa}})$ and $v_i$, $i = 1, \dots, N$ is a basis of the representation $U_{\vec{\varkappa}} \subset L_\lambda|_K$ and $v_i^*$, $i = 1, \dots, N$ is a dual basis.

As the function $\Psi_\mu$ does not depend on the choice of the basis $v_i$, $i = 1, \dots, N$ we see that $\Psi_\mu \in F_{\vec{\varkappa}}$ and its restriction to $A$ is $w_{\vec{\varkappa}} \langle s(w^*_{\vec{\varkappa}}), e^{a(x)} u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda) \rangle$, where $span\{w_{\vec{\varkappa}}\} = \tilde{U}_{\vec{\varkappa}}$, $span\{w^*_{\vec{\varkappa}}\} = \tilde{U}^*_{\vec{\varkappa}}$ and $\langle w^*_{\vec{\varkappa}}, w_{\vec{\varkappa}} \rangle = 1$. We identify $\tilde{U}_{\vec{\varkappa}}$ with $\mathbb{C}$ via $w_{\vec{\varkappa}} \to 1$ and to simplify notations we write

$$\Psi_\mu(x) = \langle w^*_{\vec{\varkappa}}, e^{a(x)} u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda) \rangle.$$

**Remark 2.8.** Such definition for $\Psi_\mu$ has a flaw. The vectors $w^*_{\vec{\varkappa}}$, $u_{\varkappa_{(1)},\varkappa_{(2)}}$ are determined up to multiplication on a constant. Hence $\Psi_\mu$ is also defined up to multiplication by a constant. We fix this constant at the end of Section 2.10.

*2.7. Eigenvalues of the radial parts of biinvariant differential operators*

The constructed function is an eigenfunction of some collection of operators. Indeed, the elements $C_r = \sum_{1 \leqslant i_1, \dots, i_r \leqslant m+n} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_r i_1}$ generate the center $\mathfrak{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$. By the Harish-Chandra theorem we have

$$C_r|_{L_\lambda} = \sum_{j=1}^{n+m} (\lambda_j^r + \text{ terms of lower degree on } \lambda) Id_{L_\lambda},$$

$$C_2|_{L_\lambda} = (\lambda, \lambda + 2\rho) Id_{L_\lambda},$$

where $\rho = \frac{1}{2}(m + n - 1, m + n - 3, \ldots, -m - n + 1)$.

**Remark 2.9.** Actually, by induction on $r$ one can prove more precise formula:

$$C_r|_{L_\lambda} = \sum_{j=1}^{n+m} ((\lambda_j + \rho_j)^r - \rho_j^r) Id_{L_\lambda},$$

but for our purposes the weaker formula is sufficient.

Using formulas from Lemma 2.2 and the previous formula, for any positive $j$ one gets

$$R_{C_2} \Psi_\mu = \left( 2(\mu + \rho_{\vec{r}}, \mu + \rho_{\vec{r}}) + \frac{C_{\vec{\varkappa}}}{2} \right) \Psi_\mu, \tag{4}$$

$$R_{C_{2j}} \Psi_\mu = 2 \left( \sum_{i=1}^{n} \mu_i^{2j} + \text{ terms of lower degree} \right) \Psi_\mu, \tag{5}$$

$$R_{C_{2j+1}} \Psi_\mu = \left( (2r+1)(\varkappa_{(1)} + \varkappa_{(2)}) \sum_{i=1}^{n} \mu_i^{2j} + \text{ terms of lower degree} \right) \Psi_\mu. \tag{6}$$

**Remark 2.10.** At the simplest case, when $n = m$, $\varkappa_{(1)} + \varkappa_{(2)} = 0$, formulas (4)–(6) are simpler:

$$R_{C_{2j}} \Psi_\mu = 2 \left( \sum_{i=1}^{n} (\mu + \rho_{\vec{r}})_i^{2j} - (\rho_{\vec{s}})_i^{2j} \right) \Psi_\mu,$$

$$R_{C_{2j+1}} \Psi_\mu = 0,$$

for any positive integer $j$.

### 2.8. *The Weyl group invariance and factorization of the spherical function*

The Weyl group $W$ of the $BC_n$ root system naturally maps onto the group $S_n$. We denote this map by $q$. The group $W$ has two independent $\mathbb{Z}_2$-characters. Indeed if $t : W \to GL(n, \mathbb{Z})$ is the tautological representation of this group, then $\chi_0(w) = det(t(w))$ and $\chi_1(w) = (-1)^{q(w)}$ form a basis of $\mathbb{Z}_2$-characters. For the character $\chi = \chi_0^{\kappa_1+\kappa_3} \chi_1^{\kappa_1+\kappa_2+\kappa_3}$ the following statement holds:

**Lemma 2.4.** *A function $f(e^{a(x)}) \in F_{\vec{\varkappa}}$ transforms under the action of $W$ by the character $\chi$. Besides, we have $f(e^{a(x+\pi i e_j)}) = (-1)^{\kappa_1} f(e^{a(x)})$ for $j = 1, \dots, n$.*

The function $\Psi_\mu(x)$ is a Laurent polynomial of $e^{x_l}$, $l = 1, \dots, n$ because $L_\lambda$ is a polynomial representation and we have. Let us denote the space of Laurent polynomials in $e^{x_l}$, $l = 1, \dots, n$ by $\mathbb{C}[P]$.

**Lemma 2.5.** *Any $f(e^{a(x)}), f \in F_{\vec{\varkappa}}$ is divisible by $\delta_{\vec{\kappa}}$ in the algebra of Laurent polynomials of $e^{x_l}$, $l = 1, \dots, n$.*

Let us denote the space of Laurent polynomials in $e^{2x_l}$, $l = 1, \dots, n$ by $\mathbb{C}[P]$. Lemmas 2.4 and 2.5 imply that the function $\Psi_\mu(x)/\delta_{\vec{\kappa}}(x)$ belongs to the space $\mathbb{C}[P]^W$ of $W$-invariant Laurent polynomials. Moreover, the following corollary holds

**Corollary 2.** (1) *$\Psi_\mu(x)/\delta_{\vec{\kappa}}(x) = \sum_{\nu \leqslant \mu} d_{\mu\nu} m_\nu(x)$, where $m_\nu$ is the orbitsum $m_\nu = \sum_{\lambda \in W\nu} e^{2(\lambda, x)}$ and $\leqslant$ is the standard $BC_n$ dominance order.*

(2) *If $\lambda \in P^{\vec{\varkappa}}$ then $\lambda = \tau(\mu + \rho_{\vec{\varkappa}})$, where $\mu \in P_+^{BC}$.*

### *2.9. The definition and properties of Heckman–Opdam's Jacobi polynomials*

Consider the operator $\widetilde{L}_{\vec{r}} = \delta_{\vec{r}}^{-1} L_{\vec{r}}^{CM} \delta_{\vec{r}}$. For this operator the following proposition holds.

**Proposition 2.1** (Heckman and Opdam [6]). *The operator $\widetilde{L}_{\vec{r}}$ maps the space $\mathbb{C}[P]^W$ of Weyl group invariant Laurent polynomials into itself. Moreover, it is triangular with respect to the basis of the orbitsums $m_\mu(x) = \sum_{\nu \in W\mu} e^{2(\nu, x)}$ i.e.*

$$\widetilde{L}_{\vec{r}} m_\mu = 4(\mu + \rho_{\vec{r}}, \mu + \rho_{\vec{r}}) m_\mu + \sum_{\nu < \mu} \alpha_{\mu\nu} m_\nu.$$

This proposition implies that one can uniquely determine the Laurent polynomial $J_\mu^{\vec{r}} = m_\mu + \sum_{\nu < \mu} s_{\mu,\nu} m_\nu$ by the condition $\widetilde{L}_{\vec{r}} J_\mu^{\vec{r}} = 4(\mu + \rho_{\vec{r}}, \mu + \rho_{\vec{r}}) J_\mu^{\vec{r}}$. Indeed, $\rho_{\vec{r}}$ is a dominant weight, hence $\nu < \mu$ implies $(\nu + \rho, \nu + \rho) < (\mu + \rho, \mu + \rho)$. Thus the operator $\widetilde{L}_{\vec{r}}$ being restricted to the finite-dimensional space *span*$\{m_\nu, \nu \geqslant \mu\}$ is diagonalizable with the distinct eigenvalues. Hence $J_\mu^{\vec{r}}$ is uniquely determined. The polynomials $J_\mu^{\vec{r}}$ are called *Heckman–Opdam's Jacobi polynomials* for the $BC_n$ root system.

It is easy to see that the operator $L_{\vec{r}}^{CM}$ is self-adjoint with respect to the standard inner product $(f, g) = \int_{A^*} f(x)\overline{g(x)}\, dx$ (here the bar means complex conjugation and $A^* = \{e^{a(x)} | \operatorname{Re} x = 0\}$). This fact implies that $J_\mu^{\vec{r}}$ is orthogonal to $J_\nu^{\vec{r}}$ if $(\mu + \rho_{\vec{r}}, \mu + \rho_{\vec{r}}) \neq (\nu + \rho_{\vec{r}}, \nu + \rho_{\vec{r}})$. In fact, an even stronger statement holds:

**Proposition 2.2** (Heckman and Opdam [6]). *The Heckman–Opdam's Jacobi polynomials $J^{\vec{r}}_{\mu}$, $\mu \in P^{BC}_{+}$ form an orthogonal basis in the space $\mathbb{C}[P]^{W}$. That is,*

$$(J^{\vec{r}}_{\mu}, J^{\vec{r}}_{\nu})_{\vec{r}} = \int_{A^*} \delta_{\vec{r}}(x)\overline{\delta_{\vec{r}}(x)} J^{\vec{r}}_{\mu} \overline{J^{\vec{r}}_{\nu}}\, dx = 0,$$

*if $\mu \neq \nu$.*

This proposition and Theorem 2.1 imply

**Corollary 3.** *The coefficient $d_{\mu\mu}$ at the expansion $\Psi_{\mu}/\delta_{\vec{\kappa}} = \sum_{\nu \leqslant \mu} d_{\mu\nu} m_{\nu}$ is not zero.*

### *2.10. The formulation of the main result*

The last corollary and Lemma 3.3 (see next section) imply $\langle v^*_{\lambda}, u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda)\rangle \neq 0$, $\langle v_{\lambda}, w_{\vec{\varkappa}}\rangle \neq 0$. Let us renormalize the function $\Psi$:

$$\widetilde{\Psi}_{\mu} = \frac{\langle s(w^*_{\vec{\varkappa}}), e^{a(x)} u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda)\rangle}{\langle s(w^*_{\vec{\varkappa}}), v_{\lambda}\rangle \langle v^*_{\lambda}, u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda)\rangle},$$

where $\lambda = \tau(\mu + \rho_{\vec{\varkappa}})$ and $v_{\lambda}$, $v^*_{\lambda}$, $\langle v_{\lambda}, v^*_{\lambda}\rangle = 1$ are the highest and lowest weight vectors for the $G$-representations $L_{\lambda}$ and $L^*_{\lambda}$, respectively. Now, the function $\widetilde{\Psi}_{\mu}$ does not depend on the choice either $w^*_{\vec{\varkappa}}$ or $u_{\varkappa_{(1)},\varkappa_{(2)}}(\lambda)$ (see the discussion at the end of Section 2.6).

The following theorem explains how to get Heckman–Opdam's Jacobi polynomials from the spherical functions. It also gives some details about the radial parts of $C_r$, $r \in \mathbb{N}$.

**Theorem 2.2.** (1) $\widetilde{\Psi}_{\mu}/\delta_{\vec{\kappa}} = J^{\vec{r}}_{\mu}$.

(2) *The radial parts $R_{C_{2i}}$, $i \in \mathbb{N}$ are pairwise commutative differential operators in $n$ variables of the form*

$$R_{C_{2i}} = 2^{1-2i} \sum_{j=1}^{n} \frac{\partial^{2i}}{\partial x_j^{2i}} + \sum_{J, |J| < 2i} a_j(x) \frac{\partial^J}{\partial x^J}.$$

**Remark 2.11.** In the case $m = n$, $\varkappa_{(1)} + \varkappa_{(2)} = 0$, the radial parts $R_{C_{2i+1}}$, $i \in \mathbb{N}$ are zero. In the general case, the radial parts $R_{C_{2i+1}}$, $i \in \mathbb{N}$ can be expressed through $R_{C_{2j}}$.

The second item of the theorem implies the complete integrability (see previous section for the definition) of the quantum Hamiltonian system defined by the Calogero–Moser operator $L^{\mathrm{CM}}_{\vec{r}}$. The first proof of the complete integrability of this system was given by Olshanetsky and Perelomov [15]. The quantum integrals $R_{C_{2i}}$

from the second part the theorem coincide with the integrals from the paper [15], because after conjugation by $\delta_{\vec{\kappa}}$ they are diagonal in the basis of Heckman–Opdam's Jacobi polynomials, with the same eigenvalues as operators from [15].

## 3. Proofs

We consider an element $g \in G$ as a $3 \times 3$ block matrix:

$$g = \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix},$$

in which the 11th, 13th, 31th and 33th blocks are $n \times n$ matrices, 12th and 21th block are $n \times (m-n)$ and $(m-n) \times n$ matrices, respectively, and 22th block is a $(m-n) \times (m-n)$ matrix. We denote these blocks by $g^{ij}$, $i,j = 1,2,3$, and the matrix elements of $g$ by $g^{ij}_{st}$, $i,j = 1,2,3$, $s = 1, \dots, n - (1 + (-1)^i)\frac{m}{2}$, $t = 1, \dots, n - (1 + (-1)^j)\frac{m}{2}$.

### 3.1. Calculation of the radial part for the Casimir element

**Proof of Theorem 2.1.** Using the formulas $E^{kl}_{ij} E^{k'l'}_{ij} = 0$, $e^{sE^{kl}_{ij}} = 1 + sE^{kl}_{ij}$ for $i \neq j$ one gets

$$\begin{aligned}
&e^{t(\sinh x_j \cosh x_i E^{11}_{ij} + \sinh x_i \cosh x_j E^{33}_{ij})} e^{a(x)} e^{sE^{31}_{ji}} e^{st \sinh(x_i + x_j)\sinh(x_i - x_j)(E^{33}_{jj} - E^{11}_{ii})} \\
&\quad \times e^{st(\sinh x_j \cosh x_j E^{31}_{jj} - \sinh x_i \cosh x_i E^{31}_{ii})} \\
&\qquad = e^{a(x)} e^{sE^{31}_{ji}} e^{t \sinh(x_j - x_i)\sinh(x_i + x_j) E^{13}_{ij}} \\
&\qquad\quad \times e^{t(\sinh x_j \cosh x_j E^{11}_{ij} + \sinh x_i \cosh x_i E^{33}_{ij})} + O(t^2) + O(s^2).
\end{aligned}$$

Substituting RHS and LHS of the last equation into the argument of a function $f \in F_{\vec{\varkappa}}$ and taking the derivative $\frac{d}{dt}$ at the point $t = 0$, one gets

$$\begin{aligned}
&(\sinh x_j \cosh x_i E^{11}_{ij} + \sinh x_i \cosh x_j E^{33}_{ij}) f(e^{a(x)} e^{sE^{31}_{ji}}) + sf(e^{a(x)} e^{sE^{31}_{ji}}) \\
&\quad \times \sinh(x_j - x_i)\sinh(x_i + x_j)(E^{33}_{jj} - E^{11}_{ii}) + s(\sinh x_j \cosh x_j D_{E^{31}_{jj}} f(e^{a(x)} e^{sE^{31}_{ji}}) \\
&\quad - \sinh x_i \cosh x_i D_{E^{31}_{ii}} f(e^{a(x)} e^{sE^{31}_{ji}})) \\
&\qquad = \sinh(x_i - x_j)\sinh(x_i + x_j) D_{E^{13}_{ij}} f(e^{a(x)} e^{sE^{31}_{ji}}).
\end{aligned} \tag{7}$$

Substituting $s=0$ to (7) and changing $i$ and $j$, we have

$$(\cosh x_j \sinh x_i E_{ji}^{33} + \sinh x_j \cosh x_i E_{ji}^{11}) f(e^{a(x)}) \\ = \sinh(x_i - x_j)\sinh(x_i + x_j) D_{E_{ji}^{31}} f(e^{a(x)}). \tag{8}$$

Taking the derivative $\frac{d}{ds}$ of formula (7) at the point $s=0$ and using (8) yields

$$\begin{aligned} D_{E_{ji}^{31}E_{ij}^{13}} f(e^{a(x)}) =& f(e^{a(x)})(E_{jj}^{33} - E_{ii}^{11}) - (\sinh x_j \cosh x_i E_{ij}^{11} + \sinh x_i \cosh x_j E_{ij}^{33}) \\ &\times \frac{(\cosh x_j \sinh x_i E_{ij}^{33} + \sinh x_j \cosh x_i E_{ji}^{11})}{\sinh^2(x_j - x_i)\sinh^2(x_i + x_j)} f(e^{a(x)}) \\ &+ \frac{\sinh x_i \cosh x_i D_{E_{ii}^{31}} - \sinh x_j \cosh x_j D_{E_{jj}^{31}}}{\sinh(x_i - x_j)\sinh(x_i - x_j)} f(e^{a(x)}). \end{aligned} \tag{9}$$

The elements $e^{a(x)} \in G$, $x \in \mathbb{C}^n$ form a commutative subgroup isomorphic to an $n$-dimensional torus ($e^{a(x+y)} = e^{a(x)}e^{a(y)}$), hence

$$\frac{\partial f}{\partial x_i}(e^{a(x)}) = (D_{E_{ii}^{31}} - D_{E_{ii}^{13}}) f(e^{a(x)}). \tag{10}$$

Substituting the formulas for the right action of $\mathfrak{k}$ on the space $\widetilde{W}_{\vec{\varkappa}} \boxtimes \widetilde{V}_{\vec{\varkappa}}$: $E_{ij}^{33}E_{ji}^{33} = \varkappa_v(\varkappa_v + 1)$, $E_{ij}^{11} = 0$, $E_{ii}^{33} = \varkappa_{(1)} + \varkappa_{(2)} - \widetilde{\varkappa}_{(1)}$, $E_{ii}^{11} = \widetilde{\varkappa}_{(1)}$, $i \neq j$ and for the left action $E_{ii}^{11} = \varkappa_{(1)}$, $E_{ii}^{33} = \varkappa_{(2)}$, $E_{ij}^{11} = E_{ij}^{33} = 0$, $i \neq j$ to the formula (9) and using (10) one gets:

$$\begin{aligned} &D_{E_{ij}^{31}E_{ji}^{13}} + D_{E_{ji}^{31}E_{ij}^{13}} + D_{E_{ij}^{13}E_{ji}^{31}} + D_{E_{ji}^{13}E_{ij}^{31}} \\ &\quad = -\varkappa_v(\varkappa_v + 1)\left(\frac{1}{\sinh^2(x_i + x_j)} + \frac{1}{\sinh(x_i - x_j)}\right) \\ &\qquad + \frac{1}{\sinh(x_i - x_j)\sinh(x_i + x_j)}\left(\sinh 2x_i \frac{\partial}{\partial x_i} - \sinh 2x_j \frac{\partial}{\partial x_j}\right). \end{aligned} \tag{11}$$

The calculation of $D_{E_{ii}^{31}E_{ii}^{13}}$ for $m \geqslant n > 1$ is absolutely the same as in the case $m = n = 1$. We make this calculation for $n = 1$.

For $f \in F_{\vec{\varkappa}}$ the following equation holds:

$$f(z) = \left(\frac{z^{13}}{z^{31}}\right)^{\frac{\widetilde{\varkappa}_{(1)} - \varkappa_{(1)}}{2}} \left(\frac{z^{11}}{z^{33}}\right)^{\varkappa_{(1)} + \frac{\varkappa_{(2)} + \widetilde{\varkappa}_{(1)}}{2}} \det(z)^{\frac{\varkappa_{(1)} + \varkappa_{(2)}}{2}} f(e^{a(x)}),$$

where $x = \operatorname{arcsinh}\left(\sqrt{\frac{z^{13}z^{31}}{\det}(z)}\right)$. Hence we have

$$f(e^{a(x)}e^{sE^{31}}e^{tE^{13}}) = \left(\frac{\sinh x + t\cosh x + st\sinh x}{\sinh x + s\cosh x}\right)^{\frac{\widetilde{\varkappa}_{(1)}-\varkappa_{(1)}}{2}} \times \left(\frac{\cosh x + s\sinh x}{\cosh x + t\sinh x + st\cosh x}\right)^{\varkappa_{(1)}+\frac{\varkappa_{(2)}+\widetilde{\varkappa}_{(1)}}{2}} f(e^{a(y)}), \quad (12)$$

where $y = \operatorname{arcsinh}(\sinh(x)\sqrt{(1+s\coth x)((1+st)+t\coth x)})$. Taking the derivative $\frac{\partial^2}{\partial s\partial t}$ of (12) at the point $s=t=0$ one gets (for any $n$)

$$D_{E_{ii}^{31}E_{ii}^{13}} = \frac{1}{4}\frac{\partial^2}{\partial x_i^2}f(e^{a(x)}) + \frac{\cosh 2x_i}{2\sinh 2x_i}\frac{\partial}{\partial x_i}f(e^{a(x)}) + \left(\frac{(\varkappa_{(2)}-\widetilde{\varkappa}_{(1)})^2}{4\cosh^2 x_i} - \frac{(\widetilde{\varkappa}_{(1)}-\varkappa_{(1)})^2}{4\sinh^2 x_i} - \frac{(\varkappa_{(1)}-\varkappa_{(2)})}{2}\left(1+\frac{(\varkappa_{(1)}-\varkappa_{(2)})}{2}\right)\right). \quad (13)$$

In the algebra $\mathfrak{k}$ there is an identity $[E_{ii}^{13}, E_{ii}^{31}] = E_{ii}^{11} - E_{ii}^{33}$. Hence we have $D_{E_{ii}^{13}E_{ii}^{31}} = D_{E_{ii}^{31}E_{ii}^{13}} + \varkappa_{(1)} - \varkappa_{(2)}$.

The calculation of $D_{E_{ij}^{23}}D_{E_{ji}^{32}}f(e^{a(x)})$ in the general case is absolutely the same as in the case $n=1$, $m=2$. For brevity we make this simplest calculation (in the general case we only have to write indices $ij$ everywhere).

We need to translate the matrix

$$e^{a(x)}e^{sE^{23}}e^{tE^{32}} = \begin{pmatrix} \cosh x & t\sinh x & \sinh x \\ 0 & 1+st & s \\ \sinh x & t\cosh x & \cosh x \end{pmatrix},$$

by the left and right action of $K$ into the form $e^{a(y)}$, for some $y\in\mathbb{C}$. That is we must find a representation of $e^{a(x)}e^{sE^{23}}e^{tE^{32}}$ in the form $e^{a(x)}e^{sE^{23}}e^{tE^{32}} = k_1e^{a(y)}k_2$, $k_1, k_2\in K$, $y\in\mathbb{C}$. Applying a function $f\in F_{\vec{\varkappa}}$ to both sides of this equation yields

$$f(e^{a(x)}e^{sE^{23}}e^{tE^{32}}) = \begin{pmatrix} u^{\frac{1}{2}} & -\frac{t\tanh x}{\cosh x(1+st)\Delta^2} & 0 \\ \frac{s}{\Delta\cosh x} & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} f(e^{a(y)}) \times u^{\frac{\varkappa_{(1)}}{2}}(1+st)^{\varkappa_{(1)}}\left(\frac{\cosh x}{\cosh y}\right)^{\varkappa_{(1)}+\varkappa_{(2)}}, \quad (14)$$

where $u = 1 + \frac{st}{(1+st)\sinh^2 x}$, $\Delta = \sqrt{\tanh^2 x + \frac{st}{(1+st)\cosh^2 x}}$, $y = \operatorname{arctanh}\Delta$.

Taking the derivative $\frac{\partial^2}{\partial s \partial t}$ of (14) at the point $s = t = 0$ and using the formulas for the right action of elements $E^{11} = \widetilde{\varkappa}_{(1)}$, $E^{22} = \varkappa_{(1)}$ one gets the formula (already in the general case)

$$D_{E^{32}_{ij} E^{23}_{ji}} f(e^{a(x)}) = \frac{\cosh x}{2 \sinh x} \frac{\partial}{\partial x_i} f(e^{a(x)}) + \left( \frac{\varkappa_{(1)} - \varkappa_{(2)}}{2} - \frac{\widetilde{\varkappa}_{(1)} - \varkappa_{(2)}}{\sinh^2 x_i} \right) f(e^{a(x)}). \tag{15}$$

Again we can calculate $D_{E^{23}_{ij}} D_{E^{32}_{ji}}$ through $D_{E^{32}_{ij}} D_{E^{23}_{ji}}$ by using the identity inside $\mathfrak{k}$.

Using the equations $D_{E^{33}_{ii}} f = \varkappa_{(2)} f$, $D_{E^{11}_{ii}} f = \varkappa_{(1)} f$, $D_{E^{33}_{ij}} f = D_{E^{11}_{ij}} f = 0$ for $i \neq j$, $f \in F_{\vec{\varkappa}}$ and (11), (13), (15) results into the formula

$$\begin{aligned} 2R_{C_2} = {} & \Delta_A + \sum_{i=1}^{n} \left( \frac{2(m-n)\cosh x_i}{\sinh x_i} + \frac{2\cosh 2x_i}{\sinh 2x_i} \right) \frac{\partial}{\partial x_i} \\ & + 2 \sum_{i=1}^{n} \sum_{j \neq i} \left( \frac{\cosh(x_i - x_j)}{\sinh(x_i - x_j)} + \frac{\cosh(x_i + x_j)}{\sinh(x_i + x_j)} \right) \frac{\partial}{\partial x_i} \\ & - 2\varkappa_v(\varkappa_v + 1) \sum_{i<j} \left( \frac{1}{\sinh^2(x_i + x_j)} + \frac{1}{\sinh^2(x_i - x_j)} \right) \\ & + \sum_{i=1}^{n} \frac{(\varkappa_{(2)} - \widetilde{\varkappa}_{(1)})^2}{\cosh^2 x_i} - \frac{(\widetilde{\varkappa}_{(1)} - \varkappa_{(1)})(\widetilde{\varkappa}_{(1)} - \varkappa_{(1)} + 2(m-n))}{\sinh^2 x_i} \\ & + (\varkappa_{(1)} + \varkappa_{(2)})^2 n + 2(m-n)\varkappa_{(1)}^2. \end{aligned}$$

Conjugating $R_{C_2}$ with $\delta_{\vec{s}}(x)$ and using a consequence of the Weyl determinant formula for the $D_n$ root system:

$$\begin{aligned} & 2 \sum_{i<j} \left( \frac{1}{\sinh^2(x_i - x_j)} + \frac{1}{\sinh^2(x_i + x_j)} \right) \\ & \quad = \sum_{i=1}^{n} \left( \sum_{j \neq i} \frac{\cosh(x_i - x_j)}{\sinh(x_i - x_j)} + \frac{\cosh(x_i + x_j)}{\sinh(x_i + x_j)} \right)^2 - (\rho_{0,1,0}, \rho_{0,1,0}), \end{aligned}$$

one gets formula (3).

### 3.2. The branching rules

Let us recall the branching rules for the inclusion $K \subset G$ (see [10]). The construction is based on the Littlewood–Richardson rule [12,13] which deals with partitions (and their diagrams). A partition is a sequence of positive integer numbers $\lambda \in \mathbb{Z}_+^r$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$, $r$ is called the length of the partition. Set $|\lambda| = \sum_{i=1}^{r} \lambda_i$. The diagram of the partition is the set of points $(i,j) \in \mathbb{Z}_+^n$ such that $1 \leqslant j \leqslant \lambda_i$.

It is more convenient to replace the points by squares (or boxes). We write $\mu \subset \lambda$ for the partitions if and only if $\mu_i \leqslant \lambda_i$ for all $i$. For a partition $\lambda$ of the length less or equal $r$ let $L_\lambda$ be the corresponding finite dimensional irreducible $GL(r)$ representation of the highest weight $\lambda$.

In these notations the branching rule has the form:

$$L_\lambda|_K = \sum_\zeta L_\zeta \boxtimes \sum_\tau c^\lambda_{\zeta\tau} L_\tau,$$

where $\lambda, \tau, \zeta$ are partitions, and $c^\lambda_{\zeta\tau}$ is a non-negative integer coefficient given by the Littlewood–Richardson rule. This coefficient is called the Littlewood–Richardson number.

### 3.3. Littlewood–Richardson rule

Let us recall the Littlewood–Richardson rule. For this purpose I need some basic combinatorial definitions. The set theoretic difference $\theta = \lambda \backslash \mu$, $\mu \subset \lambda$ is called a *skew diagram*, and $|\theta| = |\lambda| - |\mu|$. A skew diagram is a *horizontal strip* if and only if it has at most one square in each column. A *tableau* $T$ is the sequence of partitions (diagrams) $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda$ such that each of the skew diagrams $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ $(1 \leqslant i \leqslant r)$ is a horizontal strip. Graphically, $T$ may be described by the numbering each square of the skew diagram $\theta^{(i)}$ with the number $i$. The numbers inserted in $\lambda - \mu$ must increase strictly down each column and weakly from left to right along each row. The skew diagram $\lambda - \mu$ is called the shape of tableau $T$, and the sequence $(|\theta^{(1)}|, \dots, |\theta^{(r)}|)$ is called the weight of $T$.

Let $T$ be a tableau. From $T$ one can derive a word $w(T)$ by reading the symbols in $T$ from right to left in successive rows, starting with the top row. A word $w = a_1 a_2 \cdots a_N$ in the symbols $1, 2, \dots, n$ is said to be a *lattice permutation* if for $1 \leqslant r \leqslant N$ and $1 \leqslant i \leqslant n-1$, the number of occurrences of $i$ in $a_1 a_2 \cdots a_r$ is not less than the number of the occurrences of $i+1$.

$$\begin{array}{ccccc} & & \boxed{1} & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & & \\ \boxed{2} & \boxed{3} & & & \end{array}$$

For example the word $w(T)$ for the tableau from the picture is 1112132. It is an example of the lattice permutation.

**Theorem 3.1** (Littlewood and Richardson [12]). *Let $\lambda, \mu, \nu$ be partitions. Then $c^\lambda_{\mu\nu}$ is zero unless $\mu \subset \lambda$, $\nu \subset \lambda$, $|\mu| + |\nu| = |\lambda|$ and for $\mu, \nu \subset \lambda$, $|\mu| + |\nu| = |\lambda|$ it is equal to the number of tableaux $T$ of the shape $\lambda - \mu$ and weight $\nu$ such that $w(T)$ is a lattice permutation.*

The genuine definition of the Littlewood–Richardson number through Schur functions [13] implies that $c^\lambda_{\nu\mu} = c^\lambda_{\mu\nu}$.

Let $l$ be such a big integer that $\varkappa_{(i)} + l > 0$, $\widetilde{\varkappa}_{(1)} + l > 0$ and the shifted highest weights $\lambda' = \lambda + l1^r$ form a partition. Below we use the superscript prime for the shifted objects.

**Remark 3.1.** $c^{\lambda}_{\nu\mu} = c^{\lambda'}_{\nu'\mu'}$.

The last remark allows us to define the coefficient $c^{\lambda}_{\nu\mu}$, when one of the $\lambda, \nu, \mu$ is not a partition, by the formula from the remark. Below we suppose that all weight are shifted and the superscript prime is suppressed.

### 3.4. *Combinatorial proofs of Lemmas 2.1–2.3*

**Proof of Lemma 2.1.** Every irreducible representation $L_\mu$ of $GL(m-1)$, where $\mu \in P_+(GL(m-1))$ such that $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \mu_{m-1} \geqslant \lambda_m$ is contained at the restriction $L_\lambda|_{GL(m-1)}$ exactly once (see e.g.[18, p. 186]). Applying this statement $m-n$ times to $\lambda = (\widetilde{\varkappa}_{(1)} - \varkappa_{(1)})1^n + \varkappa_{(1)}1^m$ we get the first part of the lemma (e.g. $dim\, \widetilde{W}_{\vec{\varkappa}} = 1$).

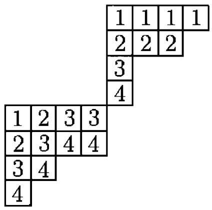
Now let us use the Littlewood–Richardson rule for the restriction from the group $K_{(1)} = GL(m)$ to the group $\widetilde{K}_{(1)} \times M = GL(n) \times GL(m-n)$. To prove the second part of the lemma one must find all partitions $\nu$ of length less or equal $n$ such that $c^{\lambda}_{\nu, \varkappa_{(1)}1^{m-n}} \neq 0$. That is one must find all fillings of $\lambda \backslash \varkappa_{(1)}1^{m-n}$ by $1, \ldots, n$ such that the result is the tableau satisfying the lattice permutation condition. The first part of the lemma says that if such filling exists then it is unique.

Let us construct this filling. Let us fill the last $\widetilde{\varkappa}_{(1)} - \varkappa_{(1)}$ squares of the $i$th row $(i = 1, \ldots, n)$ by $i$ and the first $\varkappa_{(1)}$ squares of the $(m-n+i)$th row $(i = 1, \ldots, n)$ by $i$. One can check that the resulting tableau satisfies the lattice permutation condition and has the weight $\nu = \widetilde{\varkappa}_{(1)}1^n$.

$$\begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline 1 & 2 & 2 \\ \hline 2 & & \\ \hline \end{array}$$

For example on the picture we drew the filling corresponding to the case $m = 3$, $n = 2$, $\widetilde{\varkappa}_{(1)} = 3, \varkappa_{(2)} = 1$.

To prove Lemma 2.2 we must calculate $c^{\lambda}_{\varkappa_{(1)}1^m, \varkappa_{(2)}1^n}$. Let $T = \{\varkappa_{(1)}1^m = \lambda^{(0)} \subset \cdots \subset \lambda^{(n)} = \lambda\}$ be a tableau contributing to the Littlewood–Richardson number $c^{\lambda}_{\varkappa_{(1)}1^m, \varkappa_{(2)}1^n}$. Then all the squares in the $i$th row are labeled by $i$, and the $(m+i)$th row may contain only the symbols $i, \ldots, n$ $(1 \leqslant i \leqslant n)$. Also the horizontal strip condition implies that $T$ cannot contain any label at the $(n+j)$th row $(j = 1, \ldots, m-n)$. Hence it implies $\lambda_{n+j} = \varkappa_{(1)}$. We use following notations for the labeling of the last $n$ rows: the number of occurrences of the symbol $i$ at the $(m+j)$th row is equal to $\mu^i_j$.



The drawn tableau $T$ corresponds to the case $n = 4$, $m = 4$, $\varkappa_{(1)} = 4$, $\varkappa_{(2)} = 1$. The diagram $\lambda$ has the shape : $\lambda = (8, 7, 5, 5, 4, 4, 2, 1)$. The labeling of the last $n$ rows encoded by $\mu$: $\mu_1^1 = 1$, $\mu_1^2 = 1$, $\mu_1^3 = 2$, $\mu_1^4 = 0$; $\mu_2^2 = 1$, $\mu_2^3 = 1$, $\mu_2^4 = 2$; $\mu_3^3 = 1$, $\mu_3^4 = 1$; $\mu_4^4 = 1$. The word $w(T)$ is a lattice permutation.

**Lemma 3.1.** *If* $c^{\lambda}_{\varkappa_{(1)}1^m, \varkappa_{(2)}1^n} \neq 0$ *then*

$$\lambda_j + \lambda_{m+n+1-j} = \varkappa_{(1)} + \varkappa_{(2)}, \quad (j = 1, \dots, n), \tag{16}$$

$$\lambda_{n+i} = \varkappa_{(1)} \quad (j = 1, \dots, m-n). \tag{17}$$

$$\lambda_n \geqslant \varkappa_{(1)} \tag{18}$$

*In this case* $c^{\lambda}_{\varkappa_{(1)}1^m, \varkappa_{(2)}1^n} = 1$ *and the labeling of the corresponding tableau is given by formula*:

$$\mu_j^i = \lambda_{i-j} - \lambda_{i-j+1}, \tag{19}$$

*where* $i \geqslant j$ *and* $\lambda_0 = \varkappa_{(1)} + \varkappa_{(2)}$.

**Proof.** Let us remark that the reasoning before the lemma proves formula (17). Hence the $(n+j)$th row $(j = 1, \dots, m-n)$ plays no role and we consider only the case $n = m$ in the proof.

We prove the claim by the induction in $n$.

For $n = 1$ the claim is obvious.

Let $T = \{\lambda^{(0)} \subset \cdots \subset \lambda^{(s+1)}\}$ be a tableau contributing to $c^{\lambda^{(s+1)}}_{\varkappa_{(1)}1^{s+1}, \varkappa_{(2)}1^{s+1}}$ and $\widetilde{T}$ is the tableau obtained from $T$ by deleting the boxes with $s+1$. Tableau $\widetilde{T}$ contains only the symbols $1, \dots, s$ and is of the shape $\widetilde{\lambda} \backslash \varkappa_{(1)}1^s$ for partition $\widetilde{\lambda}$, and $\widetilde{\lambda}_i = \lambda_i$ $(i = 1, \dots, s)$. Furthermore, $\widetilde{T}$ contributes to the Littlewood–Richardson number $c^{\widetilde{\lambda}}_{\varkappa_{(1)}1^s, \varkappa_{(2)}1^s}$. Hence for $\widetilde{T}$ (16)–(19) hold by the induction hypothesis. That is, the numbers $\mu_j^i$, $s \geqslant i \geqslant j$, for given $\lambda$ are uniquely determined by (19) and we only need to find $\mu_j^{s+1}$, $j = 1, \dots, s+1$. The horizontal strip condition and the induction hypothesis imply

$$\sum_{i=1}^{s+1} \mu_j^i \leqslant \sum_{i=1}^{s} \mu_{j-1}^i = \lambda_0 - \lambda_{s+2-j},$$

for $j = 2, \dots, s+1$. The induction hypothesis for $j = 1, \dots, s+1$, implies

$$\sum_{i=1}^{s+1} \mu_j^i = \widetilde{\lambda}_{s+j} + \mu_j^{s+1} = \varkappa_{(1)} + \varkappa_{(2)} - \widetilde{\lambda}_{s+1-j} + \mu_j^{s+1}$$
$$= \varkappa_{(1)} + \varkappa_{(2)} - \lambda_{s+1-j} + \mu_j^{s+1},$$

where $\widetilde{\lambda}_{2s} = 0$. Hence for $j = 2, \dots, s+1$ we have

$$\lambda_{s+2-j} \leqslant \lambda_{s+1-j} - \mu_j^{s+1}. \tag{20}$$

The lattice permutation condition for $T$ implies

$$\lambda_{s+1} + \mu_1^{s+1} \geqslant \lambda_s. \tag{21}$$

Adding these $s+1$ inequalities, one gets $\sum_{j=1}^{s+1} \lambda_j + \sum_{j=1}^{s+1} \mu_j^{s+1} \leqslant \sum_{j=0}^{s} \lambda_j$, but the weight condition for $T$ implies that the last inequality is an equality. Hence (20), (21) are also equalities and they imply (19). Thus we proved that $c^{\lambda}_{\varkappa_{(1)}1^{s+1}, \varkappa_{(2)}1^{s+1}} \neq 0$ implies (19) for $n = s+1$. One can easily check that $T$ defined by (19) contributes to the Littlewood–Richardson number $c^{\lambda}_{\varkappa_{(1)}1^{s+1}, \varkappa_{(2)}1^{s+1}}$.

Lemma 3.1 is equivalent to Lemma 2.2.

Let us denote by the symbols $v$ and $w$ the partitions $v = (\varkappa_{(1)} + \varkappa_{(2)} - \widetilde{\varkappa}_{(1)})1^n + \varkappa_v n e_1$, where $e_1 = (1, 0, \dots, 0)$, and $w = \widetilde{\varkappa}_{(1)}1^m + \varkappa_{(1)}1^n$.

**Lemma 3.2.** *If $\lambda$ is a partition of the length less or equal $n+m$ such that equalities* (16) *and inequalities*

$$\lambda_n \geqslant \widetilde{\varkappa}_{(1)}, \tag{22}$$

$$\lambda_i - \lambda_{i+1} \geqslant \varkappa_v, \tag{23}$$

*where $i = 1, \dots, n-1$, hold. Then $c^{\lambda}_{w,v} = 1$ and the labeling of the corresponding tableau is given by the formulas*:

$$\mu_j^i = \lambda_{i-j} - \lambda_{i-j+1}, \quad \text{for } i \geqslant j > 1, \tag{24}$$

$$\mu_1^i = \lambda_{i-1} - \lambda_i - \varkappa_v, \quad \text{for } i > 1, \tag{25}$$

$$\mu_1^1 = (n-1)\varkappa_v + \varkappa_{(1)} - \lambda_1, \tag{26}$$

*where $\lambda_0 = \varkappa_{(1)} + \varkappa_{(2)}$ and $\mu_j^i = 0$ for $i < j$.*

**Proof.** Let $\lambda$ be a partition of the length less or equal $m+n$, satisfying (24), (25). Let $T$ be a tableau of the shape $\lambda\backslash w$ such that the $i$th row is filled by the symbol $i$ ($i=1,\dots,n$) and the $(m+j)$th string contains $\mu^i_j$ symbols $i$, where $\mu^i_j$ are given by formulas (24)–(26). One can check that this tableau contributes to $c^\lambda_{w,v}$.

Now we will prove that if $\lambda$ satisfies the conditions from the lemma then $c^\lambda_{w,v}=1$, and Eqs. (24)–(26) hold. We will do it by the induction. There is no difference in reasonings in the case $m=n$ and in the case $m>n$, and we consider only the first case. In this case $w=\widetilde{\varkappa}_{(1)}1^n$.

For $n=2$ the claim is obvious.

Now let $T=\{\lambda^{(0)}\subset\cdots\subset\lambda^{(s+1)}=\lambda\}$ be a skew tableau satisfying inequalities (22)–(23) for $n=s+1$ which contributes to $c^\lambda_{(\widetilde{\varkappa}_{(1)})1^{s+1},v'}$, where $v'=s\varkappa_v e_1+(\varkappa_{(1)}+\varkappa_{(2)}-\widetilde{\varkappa}_{(1)})1^{s+1}$. Let us remove $\varkappa_v$ symbols 1 from the $(s+2)$th row of $T$, delete the $(s+1)$th and $2(s+1)$th rows and all boxes with $s+1$ from $T$. Then one gets a tableau $\widetilde{T}$ of some skew shape $\widetilde{\lambda}\backslash(\widetilde{\varkappa}_{(1)})1^s$, which contributes to the Littlewood–Richardson number $c^{\widetilde{\lambda}}_{(\widetilde{\varkappa}_{(1)})1^s,v''}$, $v''=\varkappa_v(s-1)e_1+(\varkappa_{(1)}+\varkappa_{(2)}-\widetilde{\varkappa}_{(1)})1^s$. Moreover, for $\widetilde{T}$ inequalities (22)–(23) hold, hence by the induction hypothesis equations (24)–(26) hold for $\widetilde{T}$. Thus we found $\mu^j_i$, $i\leqslant s$ for $T$, and we only need to find $\mu^{s+1}_j$. But we know $\lambda_i$, $i=1,\dots,2(s+1)$ and $\mu^j_i$, $i\leqslant s$, hence we can calculate $\mu^{s+1}_j$.

Remark 3.1 and Lemma 3.2 imply Lemma 2.3.

### *3.5. Proof of Lemma 2.4 and the asymptotic estimate*

**Proof of Lemma 2.4.** Let $w\in W$ to be an element of the Weyl group and $\theta_w\in\widetilde{K}_{(1)}\times K_{(2)}$ such that $\theta^{11}_w=t(w)$, $\theta^{33}_w=\hat{t}(q(w))$, here $\hat{t}$ is the standard embedding $S_n\hookrightarrow GL(n,\mathbb{Z})$. Then for $f\in F_{\vec{\varkappa}}$:

$$\begin{aligned}f(e^{a(w(x))}) &= f(\theta_w e^{a(x)}\theta_{w^{-1}}) = \theta^w f(e^{a(x)})\chi_0^{\varkappa_{(1)}}\chi_1^{\varkappa_{(2)}}(w)\\ &= \chi_0^{\widetilde{\varkappa}_{(1)}}\chi_1^{\varkappa_{(2)}+\varkappa_{(1)}-\widetilde{\varkappa}_{(1)}-\varkappa_v}(w)f(e^{a(x)})\chi_0^{\varkappa_{(1)}}\chi_1^{\varkappa_{(2)}}(w)\\ &= \chi_0^{\kappa_1+\kappa_3}\chi_1^{\kappa_1+\kappa_2+\kappa_3}(w)f(e^{a(x)}),\end{aligned}$$

here the third equality follows from Lemma 2.1 and the fact that $q(w)$ acts on $\widetilde{V}_{\vec{\varkappa}}$ by $det(\hat{t}(q(w)))^{\varkappa_{(2)}+\varkappa_{(1)}-\widetilde{\varkappa}_{(1)}-\varkappa_v}$.

The element $e^{a(\pi i e_j)}$ belongs to the subgroup $K$, hence

$$f(e^{a(x+\pi i e_j)}) = f(e^{a(x)}e^{a(\pi i e_j)}) = (-1)^{\varkappa_{(1)}+\varkappa_{(2)}}f(e^{a(x)}) = (-1)^{\kappa_1}f(e^{a(x)}).$$

We can estimate the asymptotic behavior at the infinity of a matrix element of the representation $L_\lambda$. One says that an asymptotic estimate $f(x) \lesssim g(x)$ holds in the sector, $x_1 > x_2 > \cdots > x_n$ if for any $y$ from the sector the limit $\lim_{t\to+\infty} \frac{f(ty)}{g(ty)}$ is finite.

**Lemma 3.3.** *Let $v \in L_\lambda$, $u \in L^*_\lambda$, then at the sector $x_1 > x_2 > \cdots > x_n$ we have an asymptotic estimate*

$$\langle u, e^{h(x)} v \rangle \lesssim \prod_{i=1} e^{x_i(\lambda_i - \lambda_{m+n+1-i})}. \tag{27}$$

*Moreover*

$$\lim_{t\to+\infty} \langle u, e^{a(x)} v \rangle \prod_{i=1} e^{-x_i(\lambda_i - \lambda_{m+n+1-i})} = \langle u, v_\lambda \rangle \langle v, v^*_\lambda \rangle, \tag{28}$$

*where $v_\lambda$ and $v^*_\lambda$ are the highest and lowest weight vectors of $L_\lambda$ and $L^*_\lambda$, respectively, and $\langle v_\lambda, v^*_\lambda \rangle = 1$.*

**Proof.** We use notations from the Section 2.1. There is a Cartan subgroup $H \subset G$, $A = \exp(\mathfrak{p}) \cap H$ and $e^{a(x)} = h(x, 0, -x)$. The highest weight of $L_\lambda$ with respect to $H$ is equal to $\lambda$, and all other extremal weights are of the form $w(\lambda)$, $w \in W$. Obviously, for proving the claim it is enough to prove the estimate for the case when $u, v$ are extremal weight vectors. If $u, v$ are the extremal weight vectors then $\langle u, e^{a(x)} v \rangle \sim \prod_{i=1} e^{(w(\lambda)_i - w(\lambda)_{m+n+1-i})x_i}$, for $x \sim \infty$, and obviously in the sector $x_1 > \cdots > x_n$ the asymptotic estimate (27) holds. $\square$

### 3.6. Proof of Lemma 2.5

For $f \in F_{\vec{\varkappa}}$ the equality $f(e^{a(x)}) = f(s(x))$ holds, where $s(x) \in G$ such that $s(x)^{11} = s(x)^{33} = 1$, $s(x)^{13} = s(x)^{31} = diag(z_1, \ldots, z_n)$, $s(x)^{22} = 1$, $s(x)^{23} = s(x)^{32} = s(x)^{21} = s(x)^{12} = 0$, $z_i = \tanh x_i$, $i = 1, \ldots, n$.

**Lemma 3.4.** *For any $1 \leqslant i < j \leqslant n$ the function $(E^{11}_{ij} \pm E^{22}_{ij})^m \frac{f(s(x))}{(z_i \pm z_j)^m}$ is regular at the generic point $x$ such that* $\sinh(x_i \pm x_j) = 0$.

**Proof.** For $f \in F_{\vec{\kappa}}$ the following equation holds:

$$e^{y(E^{11}_{ij} + E^{33}_{ij})} f(s(x)) = f(e^{y(E^{11}_{ij} + E^{33}_{ij})} s(x) e^{-y(E^{11}_{ij} + E^{33}_{ij})}) = f(m(x, y)),$$

where $m^{11}(x, y) = m^{33}(x, y) = 1$, $y \in \mathbb{C}$ and

$$\begin{aligned} m^{13} = m^{31} = e^{yE_{ij}} z e^{-yE_{ij}} &= (1 + yE_{ij}) z (1 - yE_{ij}) \\ &= z + y[E_{ij}, z] = z + y(z_i - z_j) E_{ij}, \end{aligned}$$

where $z = diag(z_1, \dots, z_n)$. Hence the function $e^{\frac{t(E_{ij}^{11}+E_{ij}^{33})}{z_i - z_j}} f(s(x))$ is regular at the generic point of $z_i = z_j$. By taking the derivative $\frac{d}{dt}$ at the point $t = 0$ one gets the claim for $m = 1$. Iterating this procedure we obtain the proof in the case of the minus sign. In case of the plus sign one can proceed analogously by considering $e^{y(E_{ij}^{11}-E_{ij}^{33})} f(s(x)) = f(e^{y(E_{ij}^{11}-E_{ij}^{33})} s(x) e^{-y(E_{ij}^{11}-E_{ij}^{33})})$. $\square$

**Proof of Lemma 2.5.** It is easy to see that all the weight subspaces (with respect to action of $T$) of $\widetilde{W}_{\vec{\varkappa}} \boxtimes V_{\vec{\varkappa}}$ are one dimensional. Hence for $v \in \widetilde{W}_{\vec{\varkappa}} \boxtimes \widetilde{V}\vec{\varkappa}$, we have $(E_{ij}^{11} \pm E_{ij}^{33})^l v = (\pm E_{ij}^{33})^l v \neq 0$, for $l \leqslant \kappa_2$, $i \neq j$ and zero otherwise. Using the last remark and the simple trigonometric identity $z_i \pm z_j = \frac{\sinh(x_i \pm x_j)}{\cosh x_i \cosh x_j}$, one derives from Lemma 3.4 the divisibility of $f(e^{a(x)})$ by $\delta_{0,\kappa_2,0}$.

In the rest part of the proof we consider only the case $m = n$ because the case $m > n$ is absolutely analogous. For $f \in F_{\vec{\varkappa}}$ following equation holds:

$$\begin{aligned}
\delta^{-1}_{\kappa_1,0,\kappa_3}(x) f(e^{a(x)}) &= \begin{pmatrix} \cosh^{-1} x & 0 \\ 0 & \sinh x \end{pmatrix} f(e^{a(x)}) \begin{pmatrix} 1 & 0 \\ 0 & \tanh^{-1} x \end{pmatrix} \\
&= f\left( \begin{pmatrix} \cosh^{-1} x & 0 \\ 0 & \sinh x \end{pmatrix} e^{a(x)} \begin{pmatrix} 1 & 0 \\ 0 & \tanh^{-1} x \end{pmatrix} \right) \\
&= f\left( \begin{pmatrix} 1 & 1 \\ \sinh^2 x & \cosh^2 x \end{pmatrix} \right).
\end{aligned}$$

The right-hand side of the last equation is regular at the generic point $x$ such that $\delta_{0,0,1}(x) = 0$, hence the left-hand side is also regular. Thus $f(e^{a(x)})$ is divisible by $\delta_{\kappa_1,0,\kappa_3}$.

### 3.7. *Proofs of the corollaries*

**Proof of Corollary 2.** If $\lambda = \tau(\mu + \rho_{\vec{\kappa}})$ then $\lambda_i - \lambda_{m+n+1-i} = 2(\mu + \rho_{\vec{\kappa}})_i$. In the sector $x_1 > \cdots > x_n$ $\delta_{\vec{\kappa}}$, has an asymptotic behavior $\delta_{\vec{\kappa}}(x) \sim e^{2(x, \rho_{\vec{\kappa}})}$. Hence in this sector the asymptotic estimate $\Psi_\mu(x)/\delta_{\vec{\kappa}}(x) \lesssim e^{2(x,\mu)}$ holds. Together with Lemma 2.5 it gives the proof of the first item of the corollary.

The second item immediately follows from the first one. Indeed, if for some $\mu \in \mathbb{Z}^n$, $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$, $\lambda = \tau(\mu + \rho_{\vec{\kappa}})$ belongs to $P^{\vec{\varkappa}}$ then $\Psi_\mu \neq 0$. Hence, by the previous item $\mu \geqslant 0$.

**Proof of Corollary 3.** Suppose that $d_{\mu\mu} = 0$. Then there is an expansion $\Psi_\mu / \delta_{\vec{\kappa}} = \sum_{v<\mu} c_{\mu v} J^{\vec{r}}_v$. By the definition of the Heckman–Opdam's Jacobi polynomials

$$(\Psi_\mu/\delta_{\vec{\kappa}}, \widetilde{L}_{\vec{r}}\Psi_\mu/\delta_{\vec{\kappa}})_{\vec{r}} = \left(\sum_{\nu<\mu} c_{\mu\nu}J_\nu^{\vec{r}}, \sum_{\nu<\mu}(\nu+\rho_{\vec{r}}, \nu+\rho_{\vec{r}})c_{\mu\nu}J_\nu^{\vec{r}}\right)_{\vec{r}}$$
$$=\sum_{\nu<\mu}(\nu+\rho_{\vec{r}}, \nu+\rho_{\vec{r}})c_{\mu\nu}^2(J_\nu^{\vec{r}}, J_\nu^{\vec{r}})_{\vec{r}} < (\mu+\rho_{\vec{r}}, \mu+\rho_{\vec{r}})(\Psi_\mu/\delta_{\vec{\kappa}}, \Psi_\mu/\delta_{\vec{\kappa}})_{\vec{r}}.$$

Theorem 2.1 and formula (4) yields $\widetilde{L}^{\vec{r}}\Psi_\mu/\delta_{\vec{r}} = (\mu+\rho_{\vec{r}}, \mu+\rho_{\vec{r}})\Psi_\mu/\delta_{\vec{r}}$.

### 3.8. Proof of the main theorem

**Proof of Theorem 2.2.** Corollary 3, Theorem 2.1 and formula (4) imply the first item of the theorem.

The last items follow from the fact that a $W$-invariant differential operator is uniquely determined by its action on the space $\mathbb{C}[P]^W$ of $W$-invariant polynomials (see for example [7, p. 16]). Indeed, formula (5) implies that the highest term of $R_{C_{2r}}$ has the form described at the theorem. As $C_{2r}$ are pairwise commutative, hence $R_{C_{2r}}$ are also pairwise commutative.

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